MATH 1A - FINAL EXAM SOLUTIONS

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1. *(20 points)* Use the **definition** of the integral to evaluate:

$$\int_0^1 \left(x^3 - 2\right) dx$$

You may use the following formulas:

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

<u>Preliminary work:</u>

- $f(x) = x^3 - 2$
- $a = 0$, $b = 1$, $\Delta x = \frac{1-0}{n} = \frac{1}{n}$
- $x_i = \frac{i}{n}$

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$$\begin{aligned}
\int_1^2 x^3 - 2dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n}\right)\left(\left(\frac{i}{n}\right)^3 - 2\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n}\right)\left(\frac{i^3}{n^3} - 2\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4} - \sum_{i=1}^{n} \frac{2}{n} \\
&= \lim_{n\to\infty} \frac{1}{n^4}\left(\sum_{i=1}^{n} i^3\right) - \frac{2}{n}\left(\sum_{i=1}^{n} 1\right) \\
&= \lim_{n\to\infty} \frac{1}{n^4}\left(\frac{n^2(n+1)^2}{4}\right) - \frac{2}{n}(n) \\
&= \lim_{n\to\infty} \frac{(n+1)^2}{4} - 2 \\
&= \frac{1}{4} - 2 \\
&= -\frac{7}{4}
\end{aligned}$$

Check: (not required, but useful)

$$\int_0^1 x^3 - 2dx = \left[\frac{x^4}{4} - 2x\right]_0^1 = \frac{1}{4} - 2 - (0 - 0) = -\frac{7}{4}$$

2. *(10 points)* Evaluate the following limit:

$$\lim_{n\to\infty} \frac{1}{n}\left(e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n}{n}}\right)$$

Preliminary work:

- $f(x) = e^x$
- $x_i = \frac{i}{n}$
- $a = x_0 = 0, b = x_n = 1$

Hence the limit equals to:

$$\int_0^1 e^x dx = [e^x]_0^1 = e - 1$$

3. *(40 points, 5 points each)* Find the following:

(a) $\int_{-1}^{1} \sqrt{1-x^2}dx$

Note: Don't spend too much time on this one, either you know it or you don't!

The integral represents the area of a semicircle of radius 1, hence:

$$\int_{-1}^{1} \sqrt{1-x^2}dx = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$$

(b)

$$\int \frac{1}{x^2+1} = \tan^{-1}(x) + C$$

(c) The antiderivative F of $f(x) = 3e^x + 4\sec^2(x)$ which satisfies $F(0) = 1$.

The MGAD of f is $F(x) = 3e^x + 4\tan(x) + C$. To find C, use the fact that $F(0) = 1$, so $3 + 0 + C = 1$, so $C = -2$, hence:

$$F(x) = 3e^x + 4\tan(x) - 2$$

(d)

$$\int_{0}^{1} x^3 + x^4 dx = \left[\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{1}{4} + \frac{1}{5} = \frac{9}{20}$$

(e) $g'(x)$, where $g(x) = \int_{x^2}^{e^x} \sin(t^3)dt$

Let $f(t) = \sin(t^3)$, then: $g(x) = F(e^x) - F(x^2)$, so:

$$g'(x) = F'(e^x)(e^x) - F'(x^2)(2x) = f(e^x)e^x - f(x^2)(2x) = \sin\left(e^{3x}\right)e^x - \sin\left(x^6\right)(2x)$$

(f) $\int e^x\sqrt{e^x - 1}dx$

Let $u = e^x - 1$, then $du = e^x dx$, so:

$$\int e^x\sqrt{e^x - 1}dx = \int \sqrt{u}du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}\left(e^x - 1\right)^{\frac{3}{2}} + C$$

(g) $\int_e^{e^2} \left(\frac{(\ln(x))^3}{x}\right) dx$

Let $u = \ln(x)$, then $du = \frac{1}{x}dx$, and $u(e) = \ln(e) = 1$ and $u(e^2) = 2$, so:

$$\int_e^{e^2} \left(\frac{(\ln(x))^3}{x}\right) dx = \int_1^2 u^3 du = \left[\frac{u^4}{4}\right]_1^2 = \frac{15}{4}$$

(h) The average value of $f(x) = \sin\left(x^5\right)\left(1 + e^{-x^2} + x^2\right)$ on $[-\pi, \pi]$

$$\frac{\int_{-\pi}^{\pi} \sin\left(x^5\right)\left(1 + e^{-x^2} + x^2\right)dx}{2\pi} = \frac{0}{2\pi} = 0$$

because f is an odd function.

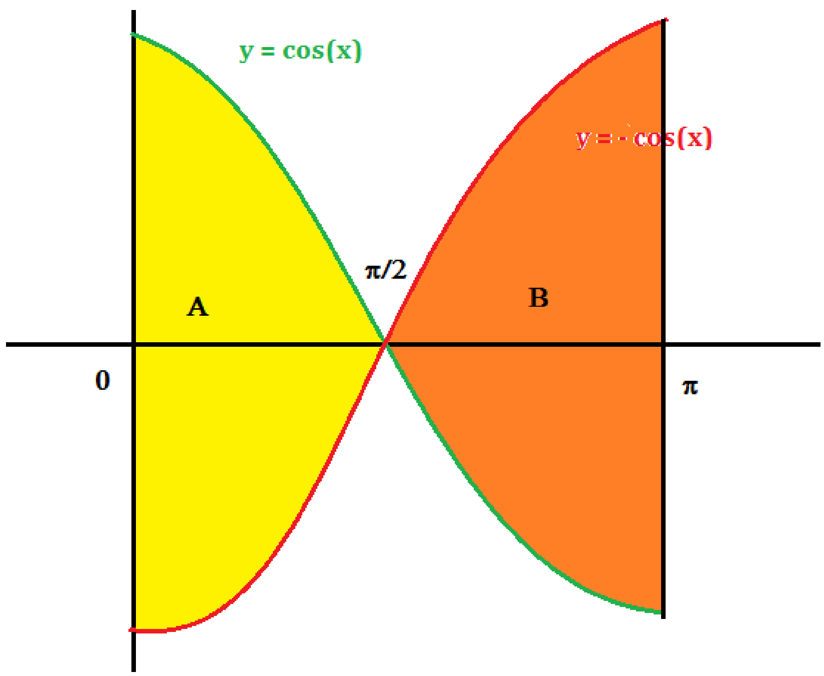
4. *(20 points)* Find the area of the region enclosed by the curves:

$$y = \cos(x) \quad \text{and} \quad y = -\cos(x)$$

from 0 to π.
Hint: It might help to notice a certain symmetry in your picture!

Picture:

1A/Math 1A Summer/Exams/Finalarea.png



Then determine the points of intersection between the two curves:

$$\begin{aligned} \cos(x) &= -\cos(x) \\ 2\cos(x) &= 0 \\ \cos(x) &= 0 \\ x &= \frac{\pi}{2} \end{aligned}$$

On $\left[0, \frac{\pi}{2}\right]$, $\cos(x)$ is above $-\cos(x)$, and on $\left[\frac{\pi}{2}, \pi\right]$, $-\cos(x)$ is above $\cos(x)$, so we'll have to figure out $A + B$ as in the picture. However, notice the symmetry! Namely, $A = B$, so all we really need to calculate is $A + B = 2A$, that is:

$$\begin{aligned}
\text{Area} =& 2\int_0^{\frac{\pi}{2}} \left(\cos(x) - (-\cos(x))\right) dx \\
=& 2\int_0^{\frac{\pi}{2}} 2\cos(x)dx \\
=& 4\int_0^{\frac{\pi}{2}} \cos(x)dx \\
=& 4\left[\sin(x)\right]_0^{\frac{\pi}{2}} \\
=& 4(1-0) \\
=& 4
\end{aligned}$$

5. *(20 points, 10 points each)* Find the following limits

(a) $\lim_{x\to-\infty} \frac{\sqrt{x^2+4}}{x}$

$$\begin{aligned}\lim_{x\to-\infty} \frac{\sqrt{x^2+4}}{x} &= \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{1+\frac{4}{x^2}}}{x} \\ &= \lim_{x\to-\infty} \frac{-x\sqrt{1+\frac{4}{x^2}}}{x} \\ &= \lim_{x\to-\infty} -\sqrt{1+\frac{4}{x^2}} \\ &= -\sqrt{1+0} \\ &= -1\end{aligned}$$

Here we used the fact that $\sqrt{x^2} = |x| = -x$, since $x < 0$.

(b) $\lim_{x\to 0^+} x^{x^2}$

1) Let $y = x^{x^2}$

2) Then $\ln(y) = x^2\ln(x)$

3)

$$\lim_{x\to 0^+}\ln(y) = \lim_{x\to 0^+} x^2\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-2}} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x\to 0^+} \frac{x^2}{-2} = 0$$

4) Hence

$$\lim_{x\to\infty} y = e^0 = 1$$

6. *(20 points, 10 points each)* Find the derivatives of the following functions

(a) $f(x) = (\sin(x))^x$

Logarithmic differentiation

1) Let $y = (\sin(x))^x$

2) Then $\ln(y) = x\ln(\sin(x))$

3) $\frac{y'}{y} = \ln(\sin(x)) + x\frac{\sin(x)}{\cos(x)}$

4)
$$y' = y\left(\ln(\sin(x)) + x\frac{\sin(x)}{\cos(x)}\right) = (\sin(x))^x\left(\ln(\sin(x)) + x\tan(x)\right)$$

(b) y', where $x^y = y^x$

Taking $\ln$s:

$$y\ln(x) = x\ln(y)$$

Differentiating and solving for y':

$$y'\ln(x) + \frac{y}{x} = \ln(y) + \frac{xy'}{y}$$
$$y'\left(\ln(x) - \frac{x}{y}\right) = \ln(y) - \frac{y}{x}$$
$$y' = \frac{\ln(y) - \frac{y}{x}}{\ln(x) - \frac{x}{y}}$$

7. *(10 points)* Find the absolute maximum and minimum of the following function on $[0, \frac{\pi}{2}]$:

$$f(x) = \sin(x) + \cos(x)$$

1) <u>Endpoints:</u> $f(0) = 0 + 1 = 1$, $f\left(\frac{\pi}{2}\right) = 1 + 0 = 1$

2) <u>Critical numbers:</u>

$$f'(x) = \cos(x) - \sin(x) = 0 \Longleftrightarrow \cos(x) = \sin(x) \Longleftrightarrow x = \frac{\pi}{4}$$

$f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

3) <u>Compare:</u> The absolute max of f is $\boxed{f\left(\frac{\pi}{4}\right) = \sqrt{2}}$ and the absolute min of f is $\boxed{f(0) = f\left(\frac{\pi}{2}\right) = 1}$

8. *(10 points)* Who's your favorite Math 1A teacher of all time???? :D
Any other goodbye words?

I hope you answered Peyam :)

Bonus 1 *(5 points)* Fill in the gaps in the following proof that the function f is not integrable on $[0,1]$:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

Step 1: Pick x_i^* such that x_i^* **is rational**. Then:

$$\begin{aligned} \int_0^1 f(x)dx &= \lim_{n\to\infty} \sum_{i=1}^n \Delta x f(x_i^*) \\ &= \lim_{n\to\infty} \sum_{i=1}^n \frac{1}{n}(0) \\ &= \lim_{n\to\infty} 0 \\ &= 0 \end{aligned}$$

Step 2: Pick x_i^* such that x_i^* **is irrational**. Then:

$$\begin{aligned} \int_0^1 f(x)dx &= \lim_{n\to\infty} \sum_{i=1}^n \Delta x f(x_i^*) \\ &= \lim_{n\to\infty} \sum_{i=1}^n \frac{1}{n}(1) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n 1 \\ &= \lim_{n\to\infty} \frac{1}{n}(n) \\ &= \lim_{n\to\infty} 1 \\ &= 1 \end{aligned}$$

Since we get two different answers for the integral, we have a contradiction. $\Rightarrow\Leftarrow$. And hence f is not integrable on $[0,1]$.

Note: See the handout 'Integration sucks!!!' for a nice discussion of this problem!

Bonus 2 *(5 points)* Another way to define $\ln(x)$ is:

$$\ln(x) = \int_1^x \frac{1}{t} dt$$

Show **using this definition only** that $\ln(e^x) = x$.

Hint: Let $g(x) = \ln(e^x) = \int_1^{e^x} \frac{1}{t} dt$. Differentiate f, simplify, and antidifferentiate. Make sure you face the issue of the constant!

Let $f(t) = \frac{1}{t}$, then $g(x) = F(e^x) - F(1)$, so:

$$g'(x) = F'(e^x)(e^x) - 0 = f(e^x)(e^x) = \frac{1}{e^x} e^x = 1$$

Since $g'(x) = 1$, we get $g(x) = x + C$. To figure out what C is, plug in $x = 0$, and we get:

$$\begin{aligned} g(0) &= 0 + C \\ \int_1^{e^0} \frac{1}{t} dt &= C \\ \int_1^1 \frac{1}{t} dt &= C \\ 0 &= C \\ C &= 0 \end{aligned}$$

Hence $g(x) = x$, so $\boxed{\ln(e^x) = x}$

Bonus 3 *(5 points)* Define the **Product integral** $\prod_a^b f(x)dx$ as follows:

If we define Δx, x_i, and x_i^* as usual, then:

$$\prod_a^b f(x)dx = \lim_{n\to\infty} \left(f(x_1^*)\right)^{\Delta x} \left(f(x_2^*)\right)^{\Delta x} \cdots \left(f(x_n^*)\right)^{\Delta x}$$

(that is, instead of summing up the $f(x_i^*)$, we just multiply them!)

Show that this is nothing new, that is, express $\prod_a^b f(x)dx$ in terms of $\int_a^b f(x)dx$

Hint: How do you turn a product into a sum?

Let $P = \prod_a^b f(x)dx$. Then:

$$\begin{aligned}
\ln(P) &= \ln\left(\lim_{n\to\infty} \left(f(x_1^*)\right)^{\Delta x} \left(f(x_2^*)\right)^{\Delta x} \cdots \left(f(x_n^*)\right)^{\Delta x}\right) \\
&= \lim_{n\to\infty} \ln\left(\left(f(x_1^*)\right)^{\Delta x} \left(f(x_2^*)\right)^{\Delta x} \cdots \left(f(x_n^*)\right)^{\Delta x}\right) \\
&= \lim_{n\to\infty} \ln\left(f(x_1^*)^{\Delta x}\right) + \ln\left(f(x_2^*)^{\Delta x}\right) + \cdots + \ln\left(f(x_n^*)^{\Delta x}\right) \\
&= \lim_{n\to\infty} \Delta x \ln\left(f(x_1^*)\right) + \Delta x \ln\left(f(x_2^*)\right) + \cdots + \Delta x \ln\left(f(x_n^*)\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \ln\left(f(x_i^*)\right) \\
&= \int_a^b \ln(f(x))dx
\end{aligned}$$

So $\ln(P) = \int_a^b \ln(f(x))dx$, so $P = e^{\int_a^b \ln(f(x))dx}$, hence:

$$\prod_a^b f(x)dx = e^{\int_a^b \ln(f(x))dx}$$